BI-TP-95/19
hep-ph/9505277
May 1995

AN ALGORITHM FOR SMALL MOMENTUM EXPANSION OF FEYNMAN DIAGRAMS

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An algorithm for obtaining the Taylor coefficients of an expansion of Feynman diagrams is proposed. It is based on recurrence relations which can be applied to the propagator as well as to the vertex diagrams. As an application, several coefficients of the Taylor series expansion for the two-loop non-planar vertex and two-loop propagator diagrams are calculated. The results of the numerical evaluation of these diagrams using conformal mapping and Padé approximants are given.

1. Introduction

Recently a new method to calculate Feynman diagrams was proposed[1]. It is based on the small momentum expansion[2], conformal mapping and construction of Padé approximants from several terms in the Taylor series of the diagram. The method was successfully applied to the evaluation of two- and three- loop diagrams[3]. As it was observed, a suitably accurate approximations to the integrals can be obtained with 20-30 coefficients in the Taylor series. The computation of two-loop vertex diagrams reveals the necessity of an efficient algorithm for the expansion of the diagrams w.r.t. external momenta.

To outline the problem, let us shortly describe the existing approach to the small momentum expansion. At present the only method of expansion is based on differentiation of the diagram w.r.t. external momenta. For the propagator type integrals the prescription for the small momentum expansion was formulated in Ref.[2]. Any coefficient in the Taylor series w.r.t. the external momentum q^2 can be obtained by applying $\Box = \frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}$ in an appropriate power to the diagram and putting the external momentum to zero.

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Supported by Bundesministerium für Forschung und Technologie.

The general expansion of a scalar 3-point function $C(q_1, q_2)$ can be written as

$$C(q_1,q_2) = \sum_{l,m,n=0}^{\infty} a_{lmn}(q_1^2)^l (q_2^2)^m (q_1 q_2)^n = \sum_{L=0}^{\infty} \sum_{l+m+n=L} a_{lmn}(q_1^2)^l (q_2^2)^m (q_1 q_2)^n. \tag{1}$$

The coefficients a_{lmn} are to be determined from a given diagram. They can be obtained by applying the differential operators $\Box_{ij} = \frac{\partial}{\partial p_{i\mu}} \frac{\partial}{\partial p_j^{\mu}}$ several times to both sides of (1). This procedure results in a system of linear equations for a_{lmn}. At a fixed L, one obtains several systems of $[L/2]+1$ equations.

Differential operators (Df's), projecting out the coefficients a_{00n} in an arbitrary space-time dimension d, are:

$$Df_{00n} = \sum_{i=1}^{[\frac{n}{2}]+1} \frac{(-4)^{1-i}\Gamma(d/2+n-i)\Gamma(d-1)}{2\Gamma(i)\Gamma(n-2i+3)\Gamma(n+d-2)\Gamma(n+d/2)} (\Box_{12})^{n-2i+2} (\Box_{11}\Box_{22})^{i-1}. \tag{2}$$

Applying Df_{00n} to $C(q_1, q_2)$ and putting the external momenta equal to zero yields the expansion coefficients a_{00n}. For the coefficients a_{l0n} the following projection operator was obtained:

$$Df_{l0n} = \frac{\Gamma(\frac{d}{2}+n)}{\Gamma(l+1)\Gamma(\frac{d}{2}+l+n)} \left(\frac{\Box_{11}}{4}\right)^l Df_{00n}. \tag{3}$$

The projection operator for arbitrary a_{lmn} is yet unknown.

An essential element of the above expansion is multiple differentiation w.r.t. multidimensional vectors. It turns out that the computer implementation of algorithms for the analytic calculation of the Taylor coefficients based on differentiation is not very effective. The computation of multiple sums with multiple differentiations in an arbitrary space time dimension d, even with the advanced possibilities existing in FORM [4], leads to significant computational difficulties.

2. Recurrence relations

In the present paper we propose a new approach to this problem, which we expect to be more suitable for the computer implementation. The algorithm for the small momentum expansion of scalar integrals can be formulated as follows:

- Firstly, the propagators with external momenta are to be expanded as

$$\frac{1}{(k_1^2 - 2k_1q_1 + q_1^2 - m_1^2)^j} = \frac{1}{(k_1^2 + q_1^2 - m_1^2)^j} \sum_{l=0}^{\infty} \frac{(l+j-1)!}{(j-1)!l!} \frac{(2k_1q_1)^l}{(k_1^2+q_1^2-m_1^2)^l}.$$

  This expansion is valid in the whole integration region since after the Wick rotation the expansion parameter is always ≤ 1.

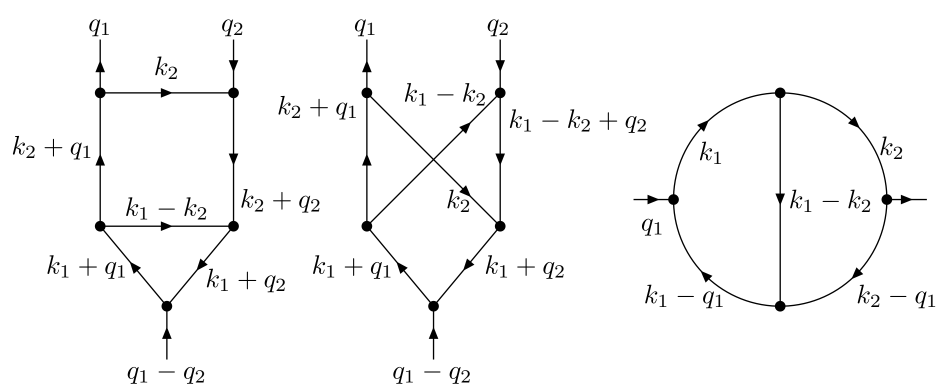
- Secondly, by using recurrence relations the external momenta are to be factored out.

- Thirdly, the remaining factors should be expanded w.r.t. the external momenta q_i^2.

- Finally, using another kind of recurrence relations, bubble integrals must be reduced to a set of master integrals and trivial ones.

The proposed algorithm does not need any explicit differentiation or solution of linear systems of equations. All required operations are efficiently implemented in many computer algebra systems. A key element of the algorithm is the recurrence relations allowing one to factorize external and integration momenta. In the present paper such recurrence relations will be presented for the two-loop case. It's not difficult to get them for higher loop diagrams. For the three-loop case they will be presented in a future publication.

We shall now describe a realization of the algorithm for the two-loop propagator and 3-point vertex diagrams. At the one-loop level the small momentum expansion is more simple, and the algorithm will be evident from the two-loop consideration. At the two- loop level there are two topologically different 3-point vertex diagrams and one propagator type topology (see $Fig.1$). We assume that the lines in $Fig.1$ correspond to arbitrary powers of the propagators.



$Fig.1$

Both vertex diagrams, after expansion of propagators w.r.t. scalar products $(k_i q_j)$, yield integrals of the type:

$$\int d^dk_1 \; d^dk_2 \; f(k_1, k_2, q_1^2, q_2^2)(k_1q_1)^{j_1}(k_1q_2)^{j_2}(k_2q_1)^{j_3}(k_2q_2)^{j_4} = v(j_1, j_2, j_3, j_4). \quad (4)$$

These satisfy the following recurrence relation:

$$(d+j_1+j_2+j_3+j_4-2)(d+j_1+j_3-3)v(j_1,j_2,j_3,j_4) =$$

$$\big\{(d+j_1+j_3+j_4-3)\left[(j_1-1)k_1^2\; q_1^2\; \mathbf{1}^- + j_2 k_1^2\;(q_1q_2)\;\mathbf{2}^- + j_3\;(k_1k_2)\;q_1^2\;\mathbf{3}^-\right]$$

$$-j_3j_4q_1^2k_2^2\mathbf{2}^+\mathbf{3}^-\mathbf{4}^- - j_4(j_4-1)k_2^2(q_1q_2)\mathbf{2}^+\mathbf{4}^-\mathbf{4}^- - j_4(j_1-1)(k_1k_2)q_1^2\;\mathbf{1}^-\mathbf{2}^+\mathbf{4}^-$$

$$+j_4(d+j_1-j_2+j_3+j_4-4)\;(k_1k_2)\;(q_1q_2)\;\mathbf{4}^-\big\}\,\mathbf{1}^-\otimes v(j_1,j_2,j_3,j_4), \qquad (5)$$

where $\mathbf{1}^{\pm}v(j_1,...)\equiv v(j_1\pm 1,...)$ etc., and the sign $\otimes$ means that the scalar products of the integration momenta in braces must be considered under the integral sign in v. Relation (5) can be derived from the following tensor formula

$$\int d^dk_1 d^dk_2 f(k_1,k_2,q_1^2,q_2^2)k_1^{\mu_1}...k_1^{\mu_N}k_2^{\nu_1}...k_2^{\nu_L} =$$

$$\frac{\Gamma(d/2)\Gamma(d-2)L!}{2^{(N+L)/2}}P_\nu\sum_{p=0}^{[\frac{L}{2}]}\frac{(L-2p+d/2-1)2^{-p}(L-2p)!}{p!\Gamma(L-p+d/2)\Gamma(L-2p+d-2)}$$

$$\sum_{r=0}^{[\frac{L}{2}]-p}\frac{\Gamma(L-2p-r+d/2-1)(-2)^{-r}}{r!(L-2p-2r)!\Gamma(\frac{d+L+N}{2}-p-r)}g^{\nu_1\nu_2}...g^{\nu_{2p+2r-1}\nu_{2p+2r}} \qquad (6)$$

$$S^{[\mu_1,...,\mu_N,\nu_{2p+2r+1},...,\nu_L]}\int d^dk_1d^dk_2 f(k_1,k_2,q_1^2,q_2^2)(k_1^2)^{\frac{N}{2}}(k_2^2)^{\frac{L}{2}}C_{L-2p}^{(d/2-1)}(\widehat{k_1k_2})$$

after contraction with q_1, q_2. Here $C_p^n(x)$ are Gegenbauer polynomials, $\widehat{k_1k_2} = (k_1k_2)/\sqrt{k_1^2k_2^2}$, S is the totally symmetric sum of products of $g's$ and P_ν means symmetrization w.r.t. the indices ν, i.e. $P_\nu g^{\nu_1\nu_2}S^{[...,\nu_3]} = (g^{\nu_1\nu_2}S^{[...,\nu_3]} + g^{\nu_1\nu_3}S^{[...,\nu_2]} + g^{\nu_2\nu_3}S^{[...,\nu_1]})/3$, etc. The integral is equal to zero if $L+N$ is odd. In the above formula $L<N$ is assumed without loss of generality.

Expansion of the self-energy diagrams will give integrals which correspond to (4) with $j_1 = j_3 = 0$. In this case the recurrence relation is very simple:

$$(d+j_2+j_4-2)v(0,j_2,0,j_4) = q_2^2\{(j_2-1)k_1^2\mathbf{2}^- + j_4(k_1k_2)\mathbf{4}^-\}\mathbf{2}^-\otimes v(0,j_2,0,j_4). \quad (7)$$

In Ref.[5] an explicit formula for $v(0,j_2,0,j_4)$ in terms of a onefold sum is given. The computer implementation of that formula, however, is less effective than the application of the recurrence relation (7). Substitution of the sum blows up expressions, producing too many similar terms at once.

A recurrence relation for the one-loop vertex diagrams is also a special case of (5) with $j_3 = j_4 = 0$:

$$(d+j_1+j_2-2)v(j_1,j_2,0,0) = k_1^2[(j_1-1)q_1^2\mathbf{1}^- + j_2\;(q_1q_2)\;\mathbf{2}^-]\mathbf{1}^-\otimes v(j_1,j_2,0,0). \quad (8)$$

Here it is assumed that the integration w.r.t. k_2 in v is omitted.

The recurrence relation (5) can also be used for evaluating anomalous dimensions of the moments of structure functions in deep inelastic scattering in the MOM renormalization scheme. In this case, one needs to calculate propagator type integrals with vertex insertions containing some additional vector ξ. The small momentum expansion of these diagrams will produce integrals of the type

$$\int d^dk_1 d^dk_2 f(k_1,k_2,q_1^2)(k_1q_1)^{j_1}(k_1\xi)^{j_2}(k_2q_1)^{j_3}, \tag{9}$$

again corresponding to the special case of (5) with $j_4=0$:

$$\begin{aligned}&(d+j_1+j_2+j_3-2)v(j_1,j_2,j_3,0) = \\ &\quad [(j_1-1)k_1^2q_1^2\mathbf{1}^- + j_2k_1^2(\xi q_1)\mathbf{2}^- + j_3(k_1k_2)q_1^2\mathbf{3}^-]\mathbf{1}^- \otimes v(j_1,j_2,j_3,0).\end{aligned} \tag{10}$$

After some modifications, the recurrence relation (5) can be used for the calculation of the Taylor coefficients in the expansion of diagrams in axial gauges.

By using relation (5) and relations derived from it by exchanging $q_1 \leftrightarrow q_2$ and $k_1 \leftrightarrow k_2$, the exponents of the scalar products can be reduced to zero. One should choose the scalar product with the smallest exponent, apply the recurrence relation until this exponent is reduced to zero and repeat this procedure until only one scalar product in some power remains. For these integrals the following explicit formula can then be used:

$$\begin{aligned}&\int f(k_1,k_2,q_1^2,q_2^2)\ (k_1q_1)^{2j_1} d^dk_1 d^dk_2 \\ &\qquad = \frac{(2j_1)!}{(d/2)_{j_1}}\left(\frac{q_1^2}{4}\right)^{j_1} \int f(k_1,k_2,q_1^2,q_2^2)(k_1^2)^{j_1} d^dk_1 d^dk_2.\end{aligned} \tag{11}$$

After that we obtain bubble-like integrals depending only on the external momenta squared:

$$\int d^dk_1\ d^dk_2\ f(k_1,k_2,q_1^2,q_2^2)\ (k_1^2)^{j_1}(k_1k_2)^{j_2}(k_2^2)^{j_3}. \tag{12}$$

The most optimal way to evaluate these integrals depends on the mass values. If some masses are the same, or one mass is zero, then it is worthwhile to expand the integrand w.r.t. q_1^2, q_2^2 and after that to calculate the bubble integrals using recurrence relations given in Refs.[2,6]. If the masses are different it will be more efficient to evaluate the integrals (12) using the recurrence relations before the expansion in q_1^2, q_2^2 is performed. In this case the presence of q_1^2, q_2^2 in the denominators can be considered as a mass shift, and therefore, the same recurrence relations can be applied.

3. Applications

To check our algorithm, we repeated the calculation of the diagram that occurs in the process $H \to \gamma\gamma$ [1]. In comparison with the old algorithm based on differentiation, the execution time reduced by a factor of about 20. For example, with

the old algorithm the evaluation of the 28-th coefficient in the Taylor series for this diagram took 5 hours CPU time on the DEC 3000. With the new algorithm, running FORM [4] on the same computer, it took 15 min to obtain the result. Further improvement in the efficiency of the computation can be achieved if the algorithm can be implemented in a multiple precision FORTRAN program [7]. In this case, as it was observed in [6,8] , the numerical approach should substantially more efficient than the analytical one.

The method proposed here was used for the evaluation of the Taylor coefficients of the nonplanar scalar vertex and propagator diagrams. In both cases all internal lines were considered to be massive. For simplicity the same mass was taken and $q_1^2 = q_2^2 = 0$ was chosen, so that the diagram depended on just one variable $q^2 = -2q_1q_2$. The evaluation of the non-planar diagram by numerical methods was complicated due to the integrable singularities [9]. It is interesting to investigate how the method proposed in[1] works in this case. The diagram has a cut for $q^2 \geq 4m^2$. After the conformal mapping we find a good convergence of the sequence of Padé approximants derived from several first Taylor coefficients. Table 1 demonstrates the quality of the convergence and contains values (up to a factor $10^{-9}/(16\pi^2)^2$) for the diagram on the cut. Errors were estimated by comparison of the [8/8] and [9/9] Padé approximants. With 18 coefficients, as one can see from Table 1, the

Table 1: Results for [9/9] on the cut $(q^2 > 4m_t^2), m = m_t = 150$ GeV

q^2/m_t^2	*nonplanar* : *vertex*	errors	*propagator* :errors		*nonplanar* : *vertex*	results
	Re	Im	Re	Im	Re	Im
4.00	1.8×10^{-8}	0.	5.7×10^{-7}	0.	0.733120	0.
4.01	7.7×10^{-9}	2.3×10^{-7}	8.0×10^{-8}	4.4×10^{-6}	0.73056733	–0.0523635
4.05	1.7×10^{-8}	7.1×10^{-8}	6.4×10^{-7}	1.1×10^{-7}	0.7204535	–0.1160966
4.50	1.3×10^{-9}	1.2×10^{-7}	1.9×10^{-6}	5.2×10^{-7}	0.61644824	–0.3349475
5.00	2.6×10^{-8}	2.7×10^{-7}	5.2×10^{-6}	2.2×10^{-6}	0.5184444	–0.430997
8.00	6.8×10^{-6}	4.9×10^{-5}	4.6×10^{-4}	7.3×10^{-5}	0.14555	–0.5460
9.0	8.4×10^{-4}	1.9×10^{-4}	6.2×10^{-4}	2.0×10^{-4}	0.0613	–0.539
10.0	9.1×10^{-3}	3.2×10^{-4}	2.5×10^{-4}	3.8×10^{-4}	0.018	–0.516
20.0	8.0×10^{-5}	3.8×10^{-4}	3.6×10^{-5}	1.5×10^{-4}	–0.2047	–0.1876
40.0	3.1×10^{-4}	1.4×10^{-3}	6.9×10^{-5}	2.0×10^{-4}	–0.1259	–0.0225
100.0	3.2×10^{-4}	3.3×10^{-3}	2.1×10^{-4}	3.4×10^{-4}	–0.0382	0.0152
200.0	3.1×10^{-3}	4.0×10^{-3}	1.8×10^{-5}	5.3×10^{-3}	–0.0125	0.0105
400.0	1.3×10^{-2}	8.3×10^{-3}	9.6×10^{-4}	3.8×10^{-3}	–0.0036	0.00511

accuracy of calculations for $q^2 < 400m^2$ is better than 1%. The convergence of the approximants below the cut is substantially better than on the cut. In Table 1 we also give results on the cut for the propagator diagram. The precision of the results for the propagator diagram is practically the same as for the vertex one. This fact can be an indication that for the diagrams with arbitrary scalar products of the external momenta, taking in (1) $L = 18 - 20$, we can achieve a similar accuracy.

Acknowledgments

I am grateful to L.Avdeev, J.Fleischer and V.Smirnov for useful discussions and

a careful reading of the manuscript and to A.Davydychev for useful discussions.

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